Framed normal curves in Euclidean space

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Abstract

A normal curve is a space curves which the position vector always lies in their normal plane. In this paper, we define framed normal curves and Frenet-type framed normal curves in Euclidean space. A framed base curve is a smooth curve with a moving frame which may have singular point [3]. We investigate the characterization of framed normal curves and reveal the relationships between framed normal curves and framed spherical curves.

2010 Mathematics Subject Classification. **53A04** 58K05

Keywords. normal curves, singular point, framed curves, framed spherical curves.

1 Introduction

The curves $\gamma : I \to \mathbb{R}^3$ for which the position vector γ always lie in their rectifying plane, are for simplicity called rectifying curves [1]. Similarly, the curves for which the position vector γ always lie in their osculating plane, are for simplicity called osculating curves; and finally, the curves for which the position vector always lie in their normal plane, are for simplicity called normal curves. K.Ilarslan and E. Nesovic investigated the notion of normal curves in Euclidean space and Minkowski space [5, 6].

Many studies have been done using Frenet frames for regular curves in classical differential geometry. If space curves have singular points, the Frenet frames of these curves cannot be constructed. However, S.Honda and M.Takahashi gave the definition of framed curves [3]. A framed curve is a smooth curve with a moving frame which may have singular point. Moreover, T.Fukunaga and M. Takahashi gave existence conditions of framed curves for smooth curves in [2]. In addition, Y.Wang, D.Pei and R.Gao defined a adapted frame for framed curves and studied framed rectifying curves in Euclidean space [8].

Inspired by the above work, in this paper, we define framed normal curves and Frenet-type framed normal curves in Euclidean space. Also, the necessary and sufficient conditions are given for a framed curve to be a framed normal curves.

2 Framed curves in Euclidean space

A framed curve in the 3-dimensional Euclidean space is a smooth space curve with a moving frame which may have singular points, in detail see [3]. Let $\gamma : I \to \mathbb{R}^3$ be a curve with singular points. The set

$$\Delta_2 = \{\mu = (\mu_1, \mu_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \mu_i, \mu_j \rangle = \delta_{ij}, \quad i, j = 1, 2\}$$

is an 3-dimensional smooth manifold. Suppose that $\mu = (\mu_1, \mu_2) \in \Delta_2$. A unit vector is defined by $\upsilon = \mu_1 \times \mu_2$.

Tbilisi Mathematical Journal Special Issue (IECMSA-2019), pp. 27–37.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 19 October 2019.
Accepted for publication: 03 January 2020.

Definition 2.1. We say that $(\gamma,\mu): I \to \mathbb{R}^3 \times \Delta_2$ is a framed curve is $\langle \gamma^{'}(s), \mu_i(s)\rangle = 0$ for all $s \in I$ and $i = 1, 2$. We also say that $\gamma : I \to \mathbb{R}^3$ is a framed base curve if there exists $\mu : I \to \Delta_2$ such that (γ,μ) is a framed curve [3].

In [2], the theorems of the existence and uniqueness for framed curves were shown as follows:

Theorem 2.2 (The Existence Theorem). Let $(l, m, n, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma,\mu) : I \to \mathbb{R}^3 \times \Delta_2$ whose associated curvature is (l, m, n, α).

Theorem 2.3 (The Uniqueness Theorem). Let (γ,μ), $(\overline{\gamma},\overline{\mu})$ be framed curves whose curvatures (l, m, n, α) and $(\overline{l}, \overline{m}, \overline{n}, \overline{\alpha})$ coincide. Then (γ,μ) and $(\overline{\gamma},\overline{\mu})$ are congruent as framed curves.

Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\upsilon(s) = \mu_1(s) \times \mu_2(s)$. The Frenet-Serret type formula is given by

$$\begin{pmatrix} \upsilon^{'}(s) \\ \mu_1^{'}(s) \\ \mu_2^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s) \\ n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \upsilon(s) \\ \mu_1(s) \\ \mu_2(s) \end{pmatrix}.$$

Here, $l(s) = \langle \mu_1^{'}(s), \mu_2(s)\rangle$, $m(s) = \langle \mu_1^{'}(s), \upsilon(s)\rangle$ and $n(s) = \langle \mu_2^{'}(s), \upsilon(s)\rangle$. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that:

$$\gamma^{'}(s) = \alpha(s)\upsilon(s).$$

In addition, s_0 is a singular point of the framed curve γ if and only if $\alpha(s_0) = 0$.

In order to obtain generalized vectors for framed curves and to get a frame similar to the Frenet-Serret frame,

$$\begin{pmatrix} \overline{\mu_1}(s) \\ \overline{\mu_2}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \mu_1(s) \\ \mu_2(s) \end{pmatrix}$$

is defined where $(\mu_1(s), \mu_2(s)) \in \Delta_2$ and $\theta(s)$ is a smooth function. Consequently, $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ also a framed curve. Let $\theta : I \to \mathbb{R}$ be a smooth function that satisfies $m(s)\sin\theta(s) = -n(s)\cos\theta(s)$. Assume that $m(s) = -p(s)\cos\theta(s), n(s) = p(s)\sin\theta(s)$, then we have:

$$\overline{\upsilon}(s) = \upsilon(s),$$

$$\begin{aligned}
\upsilon^{'}(s) &= -m(s)\mu_1(s) - n(s)\mu_2(s) = p(s)[\cos\theta(s)\mu_1(s) - \sin\theta(s)\mu_2(s)], \\
&= p(s)\overline{\mu_1}(s), \\
\overline{\mu_1}^{'}(s) &= -(l(s) - \theta^{'}(s))\sin\theta(s)\mu_1(s) \\
&\quad + (l(s) - \theta^{'}(s))\cos\theta(s)\mu_2(s) + (m(s)\cos\theta(s) - n(s)\sin\theta(s))\upsilon(s), \\
&= -p(s)\upsilon(s) + (l(s) - \theta^{'}(s))\overline{\mu_2}(s), \\
\overline{\mu_2}^{'}(s) &= -(l(s) - \theta^{'}(s))\cos\theta(s)\mu_1(s) \\
&\quad + (l(s) - \theta^{'}(s))\sin\theta(s)\mu_2(s) + (m(s)\sin\theta(s) + n(s)\cos\theta(s))\upsilon(s), \\
&= -(l(s) - \theta^{'}(s))\overline{\mu_1}(s).
\end{aligned} \tag{2.1}$$

By simple calculations, we find

$$\begin{pmatrix} \upsilon'(s) \\ \overline{\mu_1}'(s) \\ \overline{\mu_2}'(s) \end{pmatrix} = \begin{pmatrix} 0 & p(s) & 0 \\ -p(s) & 0 & q(s) \\ 0 & -q(s) & 0 \end{pmatrix} \begin{pmatrix} \upsilon(s) \\ \overline{\mu_1}(s) \\ \overline{\mu_2}(s) \end{pmatrix}.$$

We call the vectors $\upsilon(s), \overline{\mu_1}(s), \overline{\mu_2}(s)$ the generalized tangent vector, the generalized principle normal vector, and the generalized binormal vector of the framed curve, respectively, where

$$p(s) = \|\upsilon'(s)\| > 0$$

and

$$q(s) = l(s) - \theta'(s).$$

The functions $(p(s), q(s), \alpha(s))$ are referred to as the framed curvature of $\gamma(s)$ [8].

In general, the moving frame of a framed curve does not have geometric meaning. However, we can consider a moving frame with geometric meaning under a certain condition.

Definition 2.4 (**Frenet-type framed curve**). We say that $\gamma : I \to \mathbb{R}^3$ is a Frenet-type framed base curve if there exist a regular spherical curve $\mathcal{T} : I \to S^2$ and a function $\alpha : I \to \mathbb{R}$ such that $\gamma'(s) = \alpha(s)\mathcal{T}(s)$ for all $s \in I$. Then we call $\mathcal{T}(s)$ a unit tangent vector and $\alpha(s)$ a speed function of $\gamma(s)$ [4].

Then we have an orthonormal frame for Frenet-type framed curves such that

$$\{\mathcal{T}(s), \mathcal{N}(s), \mathcal{B}(s)\} = \{\mathcal{T}(s), \frac{\mathcal{T}'(s)}{\|\mathcal{T}'(s)\|}, \mathcal{T}(s) \times \mathcal{N}(s)\}.$$

Then we have the following Frenet-Serret type formula:

$$\begin{pmatrix} \mathcal{T}'(s) \\ \mathcal{N}'(s) \\ \mathcal{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{T}(s) \\ \mathcal{N}(s) \\ \mathcal{B}(s) \end{pmatrix},$$

where

$$\kappa(s) = \|\mathcal{T}'(s)\|, \quad \tau(s) = \frac{det(\mathcal{T}(s), \mathcal{T}'(s), \mathcal{T}''(s))}{\|\mathcal{T}'(s)\|^2}.$$

We can easily check that $(\gamma, \mathcal{N}, \mathcal{B}) : I \to \mathbb{R}^3 \times \Delta_2$ is a framed curve, so that we can apply the theory of framed curves.

3 Framed normal curves

In this section, the framed normal curves are defined, and we investigate their characterizations.

Definition 3.1. Let $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve. We call γ a framed normal curve if its position vector γ satisfies:

$$\gamma(s) = \lambda(s)\overline{\mu_1}(s) + \varepsilon(s)\overline{\mu_2}(s)$$

for some functions $\lambda(s)$ and $\varepsilon(s)$.

Some properties of the framed normal curves are shown in the following theorems.

Theorem 3.2. Let $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve in $\mathbb{R}^3$ with $q(s) \neq 0$. Then γ is a framed normal curve if and only if the generalized normal and binormal components of the position vector γ are respectively given by

$$\langle \gamma(s), \overline{\mu_1}(s) \rangle = -\frac{\alpha(s)}{p(s)}, \quad \langle \gamma(s), \overline{\mu_2}(s) \rangle = -\frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)^{'}.$$

Proof. First assume that $\gamma(s)$ is a framed normal curve. Therefore, we have

$$\gamma(s) = \lambda(s)\overline{\mu_1}(s) + \varepsilon(s)\overline{\mu_2}(s). \tag{3.1}$$

By taking the derivative of (3.1) and by using the corresponding Frenet equations for framed curve, we find

$$\begin{aligned} &-\lambda(s)p(s) = \alpha(s), \\ &\lambda^{'}(s) - q(s)\varepsilon(s) = 0, \\ &\lambda(s)q(s) + \varepsilon^{'}(s) = 0. \end{aligned} \tag{3.2}$$

Therefore, according to first and second equations in (3.2), we have

$$\lambda(s) = -\frac{\alpha(s)}{p(s)}, \quad \varepsilon(s) = -\frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)^{'}. \tag{3.3}$$

Consequently,

$$\gamma(s) = -\frac{\alpha(s)}{p(s)}\overline{\mu_1}(s) - \frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)^{'} \overline{\mu_2}(s).$$

Conversely, suppose that

$$\begin{aligned} \langle \gamma(s), \overline{\mu_1}(s) \rangle &= -\frac{\alpha(s)}{p(s)}, \\ \langle \gamma(s), \overline{\mu_2}(s) \rangle &= -\frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)^{'}. \end{aligned} \tag{3.4}$$

By taking the derivative of first equation of (3.4) with respect to s, we get

$$\langle \alpha(s)\upsilon(s), \overline{\mu_1}(s) \rangle + \langle \gamma(s), -p(s)\upsilon(s) + q(s)\overline{\mu_2}(s) \rangle = -\left(\frac{\alpha(s)}{p(s)} \right)^{'}. \tag{3.5}$$

Therefore, by using second equation of (3.4) in (3.5), we find

$$-p(s)\langle \gamma(s), \upsilon(s) \rangle + q(s) \left(\frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)^{'} \right) = -\left(\frac{\alpha(s)}{p(s)} \right)^{'}.$$

It follows that $\langle \gamma(s), \upsilon(s) \rangle = 0$, which means that $\gamma(s)$ is a framed normal curve. Q.E.D.

Corollary 3.3. Let $\gamma(s)$ be a framed normal curve in $\mathbb{R}^3$. If s_0 is a singular point of the framed normal curve by according to (3.2) equations, then

$$\langle \gamma(s_0), \overline{\mu_1}(s_0) \rangle = \lambda(s_0) = 0,$$

$$\langle \gamma(s_0), \overline{\mu_2}(s_0) \rangle = \varepsilon(s_0) = \frac{\alpha' s_0}{p(s_0)q(s_0)}.$$

Theorem 3.4. Let $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve in $\mathbb{R}^3$ with $q(s) \neq 0$. Then γ is congruent to a framed normal curve if and only if

$$-\frac{\alpha(s)q(s)}{p(s)} = \left(\frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)' \right)'. \tag{3.6}$$

Proof. Let us assume that $\gamma(s)$ is congruent to a framed normal curve. According to equation (3.4), we have relation (3.6). Conversely, assume that $\gamma(s)$ is a framed curve satisfying the relation (3.6). By applying Frenet equation for framed curve, we have

$$\frac{d}{ds}\left[\gamma(s) + \frac{\alpha(s)}{p(s)}\overline{\mu_1}(s) + \frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)' \overline{\mu_2}(s) \right] = 0.$$

Consequently, γ is a congruent to a framed normal curve. Q.E.D.

Theorem 3.5. Let $\gamma(s)$ be a framed curve with $q(s) \neq 0$. Then $\gamma(s)$ lies on sphere $S^2(r)$ for $\forall s \in I$ if and only if $\gamma(s)$ is a framed normal curve.

Proof. First assume that $\gamma(s)$ is a framed base curve lying on a sphere of radius r and center m. Therefore, we have

$$< \gamma(s) - m, \gamma(s) - m >= r^2. \tag{3.7}$$

By differentiation in s, we find that

$$2\alpha \langle \upsilon, \gamma - m \rangle = 0. \tag{3.8}$$

Let us assume that $\langle \gamma - m, \upsilon \rangle = h$ for every $s \in I$. Therefore, if $\alpha h = 0$ for each $s \in I$, it is either $\alpha = 0$ or $h = 0$. If $\alpha(s) = 0$ for all $s \in I$, then γ is a point [3]. Consequently, since γ is a framed spherical curve, $h = 0$ for every $s \in I$. Even if $\alpha(s_0) = 0$ for $s = s_0$ singular point, $h(s_0) = 0$ will be provided. Therefore, we have that

$$\langle \upsilon, \gamma - m \rangle = 0, \tag{3.9}$$

for each $s \in I$. By differentiation in s of equations (3.9) two times and applying Frenet formulae, we find

$$\gamma(s) - m = -\frac{\alpha(s)}{p(s)}\overline{\mu_1}(s) - \frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)' \overline{\mu_2}(s).$$

Consequently, γ is a framed normal curve. Conversely, suppose that $\gamma(s)$ is a framed normal curve. According to Theorem 3.4, we have

$$-\frac{\alpha(s)q(s)}{p(s)} = \left(\frac{1}{q(s)} \left(\frac{\alpha(s)}{p(s)} \right)' \right)'. \tag{3.10}$$

Let us consider the vector

$$m=\gamma(s)+\frac{\alpha(s)}{p(s)}\overline{\mu_1}(s)+\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{'}\overline{\mu_2}(s).$$

Differentiating this equation with respect to s, we get $m^{'}=0$. Consequently, $m=constant$. Since (3.10) is the differential of the equation

$$\left(\frac{\alpha(s)}{p(s)}\right)^2+\left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{'}\right)^2=c=constant.$$

Therefore, we have

$$\langle\gamma-m,\gamma-m\rangle=\left(\frac{\alpha(s)}{p(s)}\right)^2+\left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{'}\right)^2=c=constant.$$

We may take $c=r^2$. Finally, it follows that image of the framed base curve $\gamma(s)$ lies on a sphere of radius r.

Q.E.D.

Theorem 3.6. Let $(\gamma,\overline{\mu_1},\overline{\mu_2}):I\to\mathbb{R}^3\times\Delta_2$ be a framed curve in $\mathbb{R}^3$ with $q(s)\neq 0$. $\gamma(s)$ is a framed normal curve if and only if the following statements are satisfied:
i.) The framed curvatures $p(s),q(s)$ and $\alpha(s)$ satisfy the following equality

$$\frac{\alpha(s)}{p(s)}=c_1\cos\left(\int q(s)ds\right)+c_2\sin\left(\int q(s)ds\right)$$

for some constant $c_1,c_2\in\mathbb{R}$
ii.) The generalized principal normal and generalized binormal component of the position vector of the framed curve are given respectively by

$$\langle\gamma(s),\overline{\mu_1}(s)\rangle=-c_1\cos\left(\int q(s)ds\right)-c_2\sin\left(\int q(s)ds\right),$$

$$\langle\gamma(s),\overline{\mu_2}(s)\rangle=-c_1\sin\left(\int q(s)ds\right)+c_2\cos\left(\int q(s)ds\right).$$

Conversely if $\gamma(s)$ is framed curve in $\mathbb{R}^3$, the framed curvature $q(s)\neq 0$ for each $s\in I$ and one of the statements (i) and (ii) hold, then γ is a framed normal curve or congruent to a framed normal curve.

Proof. First assume that $\gamma(s)$ is a framed normal curve. According to equation (3.6), we have

$$-\frac{\alpha(s)q(s)}{p(s)}=\left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{'}\right)^{'}.$$

Putting $t(s)=\dfrac{1}{p(s)}$ and $z(s)=\dfrac{1}{q(s)}$, equation (3.6) can be written as

$$-\frac{\alpha(s)t(s)}{z(s)}=\left(z(s)[\alpha(s)t(s)]'\right)'$$

If we change variables in the last equation as $x=\int\dfrac{1}{z(s)}ds$, then we have

$$\frac{d^2(\alpha t)}{dx^2}+\alpha t=0 \tag{3.11}$$

The solution of (3.11) differential equation is

$$\alpha(s)t(s)=c_1\cos x+c_2\sin x$$

where $c_1,c_2\in\mathbb{R}$. Consequently,

$$\frac{\alpha(s)}{p(s)}=c_1\cos\left(\int q(s)ds\right)+c_2\sin\left(\int q(s)ds\right) \tag{3.12}$$

Therefore, we have proved statement (i). According to equation (3.3) and (3.12), we get

$$\lambda(s)=-c_1\cos\left(\int q(s)ds\right)-c_2\sin\left(\int q(s)ds\right)$$

$$\varepsilon(s)=-c_1\sin\left(\int q(s)ds\right)+c_2\cos\left(\int q(s)ds\right).$$

Consequently, the framed normal curve is denote by

$$\begin{aligned}\gamma(s)=&\left[-c_1\cos\left(\textstyle\int q(s)ds\right)-c_2\sin\left(\textstyle\int q(s)ds\right)\right]\overline{\mu_1}(s)+\\&\left[-c_1\sin\left(\textstyle\int q(s)ds\right)+c_2\cos\left(\textstyle\int q(s)ds\right)\right]\overline{\mu_2}(s).\end{aligned} \tag{3.13}$$

Therefore, from (3.13), we have

$$\langle\gamma(s),\gamma(s)\rangle=c_1^2+c_2^2, \tag{3.14}$$

$$\langle\gamma(s),\overline{\mu_1}(s)\rangle=-c_1\cos\left(\int q(s)ds\right)-c_2\sin\left(\int q(s)ds\right), \tag{3.15}$$

$$\langle\gamma(s),\overline{\mu_1}(s)\rangle=-c_1\sin\left(\int q(s)ds\right)+c_2\cos\left(\int q(s)ds\right). \tag{3.16}$$

Therefore, we have proved statement (ii). Conversely, let us suppose that statement (i) holds. Therefore, we have

$$\frac{\alpha(s)}{p(s)}=c_1\cos\left(\int q(s)ds\right)+c_2\sin\left(\int q(s)ds\right).$$

Differentiating last equation with respect to s, we get

$$-\frac{\alpha(s)q(s)}{p(s)} = \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'.$$

By applying Frenet equations (1) for framed curves, we find

$$\frac{d}{ds}\left[\gamma(s) + \frac{\alpha(s)}{p(s)}\overline{\mu_1}(s) + \frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\overline{\mu_2}(s)\right] = 0.$$

Consequently, γ is a framed normal curve. Next, let us suppose that statement (ii) holds. Therefore, we have

$$\langle\gamma(s), \overline{\mu_1}(s)\rangle = -c_1\cos\left(\int q(s)ds\right) - c_2\sin\left(\int q(s)ds\right), \tag{3.17}$$

$$\langle\gamma(s), \overline{\mu_2}(s)\rangle = -c_1\sin\left(\int q(s)ds\right) + c_2\cos\left(\int q(s)ds\right). \tag{3.18}$$

By differentiation in s of (3.17) and using (3.18), we get

$$\langle\gamma(s), \upsilon(s)\rangle = 0. \tag{3.19}$$

Consequently, by according to (3.19), γ is a framed normal curve

Q.E.D.

Corollary 3.7. Let $\gamma(s)$ be a framed normal curve. If s_0 is a singular point of the framed normal curve, then

$$c_1\cos\left(\int q(s)ds\right) = -c_2\sin\left(\int q(s)ds\right)$$

for $s_0 \in I$.

4 Frenet-type framed normal curves

Definition 4.1. Let $(\gamma, \mathcal{N}, \mathcal{B}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve. We call γ a framed normal curve if its position vector γ satisfies:

$$\gamma(s) = \lambda(s)\mathcal{N}(s) + \varepsilon(s)\mathcal{B}(s)$$

for some functions $\lambda(s)$ and $\varepsilon(s)$.

Some properties of the Frenet-type framed normal curves are shown in the following theorems. The proofs of the Theorems are similar to the Theorem (3.3) and Theorem (3.5).

Theorem 4.2. Let $(\gamma, \mathcal{N}, \mathcal{B}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve in $\mathbb{R}^3$ with framed curvatures $\kappa(s) > 0$, $\tau(s) \neq 0$. Then γ is a framed normal curve if and only if the normal and binormal components of the position vector γ are respectively given by

$$\langle\gamma(s), \mathcal{N}(s)\rangle = -\frac{\alpha(s)}{\kappa(s)}, \quad \langle\gamma(s), \mathcal{B}(s)\rangle = -\frac{1}{\tau(s)}\left(\frac{\alpha(s)}{\kappa(s)}\right)'.$$

Theorem 4.3. Let $(\gamma, \mathcal{N}, \mathcal{B}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve in $\mathbb{R}^3$ with framed curvatures $\kappa(s) > 0$, $\tau(s) \neq 0$. Then γ is congruent to a framed normal curve if and only if

$$-\frac{\alpha(s)\tau(s)}{\kappa(s)} = \left(\frac{1}{\tau(s)}\left(\frac{\alpha(s)}{\kappa(s)}\right)'\right)'. \tag{4.1}$$

Example 4.4. (The spherical nephroid ([4]). The spherical nephroid

$$\gamma : [0, 2\pi) \to S^2 \subset \mathbb{R}^3$$

is defined by

$$\gamma(s) = \left(\frac{3}{4}\cos s - \frac{1}{4}\cos 3s, \frac{3}{4}\sin s - \frac{1}{4}\sin 3s, \frac{\sqrt{3}}{2}\cos s\right). \tag{4.2}$$

See Fig.1. Then

$$\mathcal{T}(s) = \frac{1}{2}\left(\sqrt{3}\cos 2s, \sqrt{3}\sin 2s, -1\right) \tag{4.3}$$

gives the unit tangent vector and $\alpha(s) = \sqrt{3}\sin s$. By a calculation, we get

$$\mathcal{N}(s) = (-\sin 2s, \cos 2s, 0), \tag{4.4}$$

$$\mathcal{B}(s) = \frac{1}{2}\left(\cos 2s, \sin 2s, \sqrt{3}\right). \tag{4.5}$$

Moreover, we find,

$$\kappa(s) = \|\mathcal{T}'(s)\| = \sqrt{3}$$

$$\tau(s) = \frac{det(\mathcal{T}(s), \mathcal{T}'(s), \mathcal{T}''(s))}{\|\mathcal{T}'(s)\|^2} = -1.$$

According to equations (3.1), (3.3) and (3.4), we get

$$\left(\frac{3}{4}\cos s - \frac{1}{4}\cos 3s, \frac{3}{4}\sin s - \frac{1}{4}\sin 3s, \frac{\sqrt{3}}{2}\cos s\right) = \lambda(s)\left(-\sin 2s, \cos 2s, 0\right) + \varepsilon(s)\left(\frac{\cos 2s}{2}, \frac{\sin 2s}{2}, \frac{\sqrt{3}}{2}\right).$$

Therefore, we find $\lambda(s) = -\sin s$ and $\varepsilon(s) = \cos s$. Moreover, $\lambda(s_0) = 0$ in singular point of γ. Moreover, according to theorem 4.2, we have

$$\langle \gamma(s), \mathcal{N}(s)\rangle = -\sin s \tag{4.6}$$

and

$$-\frac{\alpha(s)}{\kappa(s)} = -\sin s, \tag{4.7}$$

$$\langle \gamma(s), \mathcal{B}(s)\rangle = \cos s \tag{4.8}$$

and

$$-\frac{1}{\tau(s)}\left(\frac{\alpha(s)}{\kappa(s)}\right)' = \cos s. \tag{4.9}$$

Consequently, γ is a framed normal curve.

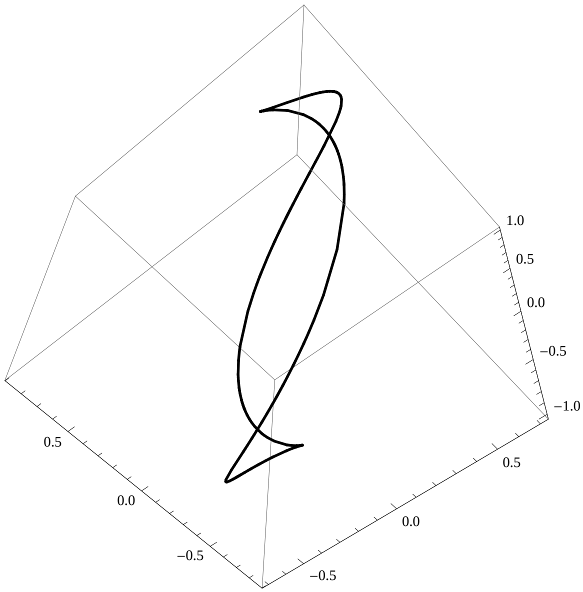

Figure 1. Spherical Nephroid is a framed normal curve.

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